Datalogisk Institut
KØBENHAVNS UNIVERSITET

Two-Connected Steiner Networks: Structural Properties

Pawel Winter and Martin Zachariasen

Technical Report no. 2004/01
ISSN: 0107-8283
CR Subject classification G.2.2

DIKU
University of Copenhagen • Universitetsparken 1
DK-2100 Copenhagen • Denmark

Two-Connected Steiner Networks: Structural Properties

Pawel Winter and **Martin Zachariasen**
Dept. of Computer Science, University of Copenhagen
Universitetsparken 1, DK-2100 Copenhagen Ø, Denmark
e-mail: {pawel,martinz}@diku.dk

January 9, 2004

Abstract

We consider the problem of constructing a minimum-length 2-connected network for a set of terminals in a graph where edge-weights satisfy the triangle inequality. As a special case of the problem, we have the Euclidean 2-connected Steiner network problem in which a set of points in the plane should be interconnected by a 2-connected network of minimum length. These problems have natural applications in the design of surviable communication networks.

In this paper we give a number of structural results for the problem. We show that all cycles must have at least four terminals, and that these terminals must be distributed in groups of two. Furthermore, we give a tight lower bound on the minimum size of any problem instance that requires a Steiner vertex.

Keywords: Survivable networks, 2-connected Steiner networks.

1 Introduction

The well-known Steiner tree problem asks for a shortest possible network spanning a set Z of terminals in the plane. The solution to the problem is a tree, referred to as a *Steiner minimal tree* (SMT). Apart from the terminals that must be spanned, the SMT may contain additional, so-called *Steiner points*, where exactly three edges meet at 120^o angles. SMTs are unions of *full Steiner trees* spanning subsets of terminals all having degree 1.

When the objective is to design low cost survivable networks, the problem of constructing 2-connected Steiner minimal networks in the plane arises. In a 2-edge-connected (resp. 2-vertex connected) Steiner network there are a least two edge-disjoint (resp. vertex-disjoint) paths between every pair of terminals. When the distance function satisfies the triangle inequality, there is no need to distinguish between 2-vertex and 2-edge connected problems since 2-edge-connected solutions will automatically be 2-vertex-connected [2].

The 2-connected Steiner network problem in the plane has been studied by Luebke and Provan [4], who proved that it is NP-hard and gave a number of structural properties. Additional structural properties for the generalized graph version where distances fulfill the triangle inequality were given by Luebke [3].

In this paper we significantly improve and generalize the results given in [4, 3]. We show that all cycles must have at least four terminals and that no cycles can consist of edges solely from full Steiner trees spanning three or more terminals. This allows us to give a tight lower bound on the number of terminals needed in any instance that requires a Steiner vertex in a minimum-length solution. Our results will be applied in a new exact algorithm that is described in an accompanying paper [7].

The paper is organized as follows. In Section 2 we formally define the problems that are considered; also we review some of the results known for these problems. In Section 3 we prove some properties for so-called chord-paths, and in Section 4 we give our main results on the distribution of terminals in cycles of Steiner networks. Lower bounds (and tight examples) on the size of Steiner networks that require Steiner vertices are given in Section 5; a discussion of applications of our results and concluding remarks are given in Section 6.

2 Preliminaries

Let $G = (V, E)$ be a complete undirected graph with a distance function defined on its vertices. For a pair of vertices $u, v \in V$, let $|uv|$ denote the distance between u and v. The distance function is assumed to fulfill the requirements of a metric, i.e., it is non-negative, symmetric and satisfies the triangle inequality. Finally, let $Z \subseteq V$ be a set of *terminals*; the remaining vertices $V \setminus Z$ are denoted *Steiner vertices*.

The *2-connected Steiner network problem* (2-SNPG) is to find subgraph $G' = (V', E')$ of G such that

- G' is 2-edge-connected
- $Z \subseteq V'$
- G' has minimum total length wrt. the distance function

Since a 2-edge-connected minimum-length network necessarily is 2-vertex-connected when the distance function is a metric [1], we use the shorthand 2-connected in the following.

This problem was studied by Monma et al. [1] — however mainly for the case when $Z = V$, that is, when all vertices should be interconnected. They proved that there always exists an optimal solution in which all terminals have degree 2 or 3, and all Steiner vertices have degree 3.

Hsu and Hu [2] and Luebke and Provan [4] considered a special case of the graph problem, namely the *Euclidean 2-connected Steiner network problem in the plane* (2-SNPP). In this problem the terminals Z are points in the plane, and the task is to construct a minimum-length 2-connected network that interconnects Z. For this problem it was proved that Steiner points are incident to three edges meeting at $120°$ angles (as for the Euclidean Steiner tree problem in the plane), and that no cycles in a minimum-length network consist entirely of Steiner points. As a consequence, a shortest network is a union of full Steiner trees (FSTs), in which all terminals are leaves and all Steiner points are interior vertices.

In this paper we study the structure of minimum-length solutions for the general graph problem (2-SNPG) — which contains 2-SNPP as a special case. More specifically, we will study the structure of minimum-length solutions in which the *total degree* of all vertices is minimized. In this way we avoid trivial degeneracies. We denote by SMN an arbitrary minimum-length 2-connected network for which the total degree of all vertices has been minimized. (For 2-SNPP an arbitrary 2-connected minimum-length network is denoted by SMNP). Luebke [3] proved that the following properties hold for any SMN:

- All vertices have degree 2 or 3, and all Steiner vertices have degree 3.

- All edges are of multiplicity one.
- No cycle is composed entirely of Steiner vertices.
- For $|Z| \geq 4$ there are no cycles with exactly three vertices.

3 Chord-path properties

Let G denote an arbitrary undirected graph. Given a cycle C and two distinct vertices u and v on C in G, a *chord-path* between u and v is a path $P(u, v)$ in G between u and v that, except from u and v, shares neither vertices nor edges with C. Note that the interior vertices in $P(u, v)$ are not required to have degree 2 in G. When $P(u, v)$ consists of a single edge, the chord-path reduces to a simple *chord edge* of C.

It is well-known [1, 3] that a SMN cannot have any chord edge: Consider a cycle C having a chord edge (u, v). Clearly, the subgraph $C \cup (u, v)$ is 2-connected. If we delete edge (u, v) from this subgraph, the total length of the network does not increase and the total degree decreases. Furthermore, since the remaining subgraph (which is C) is still 2-connected, the resulting overall network remains 2-connected [1].

As a consequence, any chord-path in a SMN must have at least two edges. The following lemma strengthens this result.

Lemma 1 *Any chord-path in a SMN must have at least* three *edges.*

Proof. Assume that there exists a cycle C with a chord-path $P(u, v)$ consisting of two edges (u, t) and (t, v). Let a and b be the neighbours of u on C, and let c and d be the neighbours of v on C (see Figure 1a). Neither vertices a and d nor vertices b and c are required to be distinct.

Assume w.l.o.g. that $|tv| \leq |tu|$. Remove the edges (t, u) and (v, c), and add the edge (t, c), as shown in Figure 1b. The network remains 2-connected and its length does not increase: $|tc| \leq |tv| + |vc| \leq |tu| + |vc|$. Both vertices u and v have reduced their degree with one, while the degree of all other vertices is unchanged. Thus we have arrived at a contradiction to the (length and degree) minimality of SMN. ■

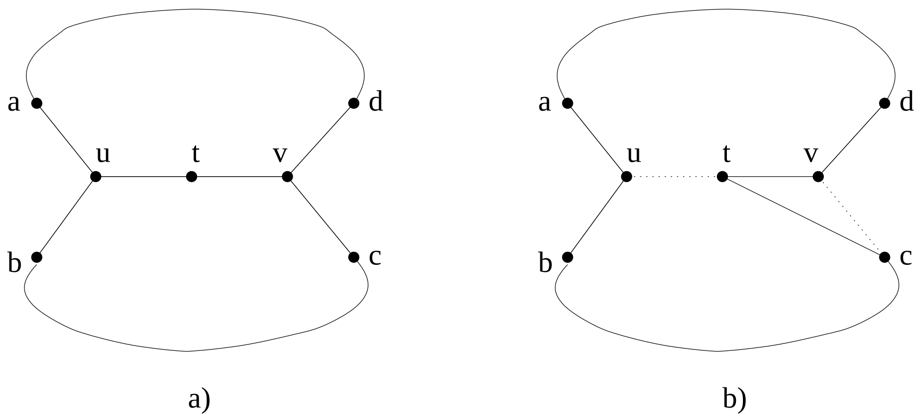


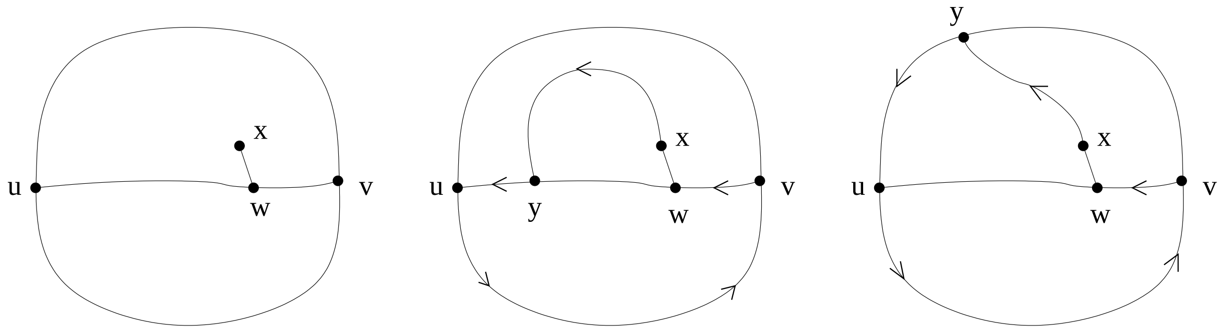
Figure 1: A chord-path consisting of two edges.

Luebke and Provan [4] showed for 2-SNPP that a SMNP cannot have a chord-path that has only Steiner points in its interior. Using Lemma 1, we can now generalize this result.

Theorem 1 *Any chord-path in a SMN must have a pair of consecutive terminals of degree 2 in its interior.*

Proof. Consider a cycle C and a path-chord $P(u, v)$ without two consecutive terminals of degree 2 in its interior. Among all such path-chords, let $P(u, v)$ be one with the *minimum* number of edges.

By Lemma 1, $P(u, v)$ has at least two interior vertices, and hence at least one interior vertex w of degree 3. Let x be the third vertex adjacent to w and not on $P(u, v)$ (Figure 2a). Edge (w, x) must be on some cycle C_w in SMN; this cycle uses one of the two edges on $P(u, v)$ incident to w, since w has degree 3. As a consequence, C_w will share at least one vertex with $C \cup P(u, v)$, and distinct from w.



a) b) c)

Figure 2: A chord-path with a degree 3 vertex.

Follow cycle C_w, starting in x and moving away from w. Let y be the first vertex on $C \cup P(u, v)$ that is encountered. Assume first that y is some vertex on $P(u, v)$. Then the subpath of $P(u, v)$ from w to y is a chord-path of another cycle in SMN, as shown in Figure 2b. This chord-path has fewer edges than $P(u, v)$. Furthermore, it has no pair of consecutive degree 2 terminals. This is a contradiction to the choice of $P(u, v)$.

Now assume that y is a vertex on C, distinct from u and v. The subpath of $P(u, v)$ from u to w is a chord-path of another cycle, as shown in Figure 2c. Again, this chord-path has fewer edges than $P(u, v)$ and has no pair of consecutive degree 2 terminals. Once again this is a contradiction to the choice of $P(u, v)$. In conclusion, chord-path $P(u, v)$ cannot exist. ■

4 Cycle properties

In this section we use the properties of chord-paths to prove a number of fundamental structural properties for cycles in solutions to 2-SNPG.

Theorem 2 *Consider a cycle C that contains a vertex of degree 3 in a SMN. Then C must have* two *pairs of consecutive terminals, both of degree 2 in SMN. Furthermore, these two terminal pairs must be separated on C by a pair vertices of degree 3.*

Proof. Let w be a vertex on C of degree 3. Let x be the third vertex adjacent to w and not on C. Edge (w, x) must be on some cycle C_w in SMN; this cycle uses one of the two edges on C incident to w, since w has degree 3. As a consequence, C_w will share at least one vertex with C, and distinct from w. Follow cycle C_w, starting in x and moving away from w. Let y be the first vertex on C that is encountered, and define P_1 to be the path from w to y following cycle C_w.

Let P_2 and P_3 denote the two edge-disjoint paths from y to w following cycle C. Note that the paths P_1, P_2 and P_3 share no vertices nor edges except from their endpoints. Consider the cycle $P_1 \cup P_2$. The path P_3 is a chord-path of this cycle and by Theorem 1 it has a pair of consecutive terminals of degree 2 in SMN. Now consider the cycle $P_1 \cup P_3$. The path P_2 is a chord-path of this cycle and must also have a pair of consecutive terminals of degree 2 in SMN. The theorem follows. ■

Consider an instance of 2-SNPG with at least four terminals. Either the SMN for this problem is a simple cycle through all terminals, or every cycle has a vertex of degree 3. In both cases we get the following:

Corollary 1 *If the total number of terminals in a SMN is at least four, then every cycle in SMN has at least* four *terminals of degree 2.*

A SMN is a union of full Steiner trees (FSTs) [4]. Consider a cycle C in a SMN in which every edge is from an FST spanning at least three terminals (Figure 3). Thus every terminal in C has two Steiner vertices as neighbours, that is, cycle C has no pair of consecutive terminals of degree 2 as required by Theorem 2. We have the following corollary:

Corollary 2 *No cycle in a SMN has edges solely from full Steiner trees spanning three or more terminals.*

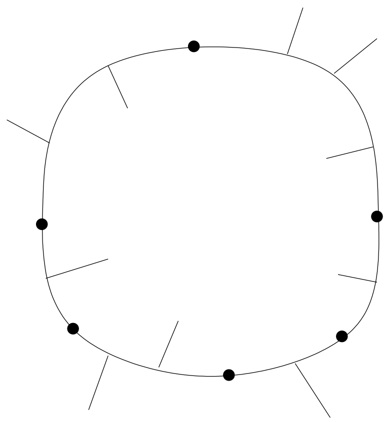


Figure 3: A cycle with edges solely from FSTs spanning three or more terminals. In this example, the cycle contains 6 terminals; the remaining vertices on the cycle are Steiner vertices in FSTs spanning three or more terminals.

5 Smallest networks with Steiner vertices

In this section we will show that a SMN cannot have vertices of degree 3 unless it contains at least 6 terminals. We will also show that this bound is tight for 2-SNPG. We conjecture that the smallest number of terminals needed for an SMNP (i.e., in the Euclidean plane) to have vertices of degree 3 is 8. We also give a problem instance with 8 terminals where the SMNP in fact has 2 Steiner points.

Throughout this section, we extensively use the fact that cycles of SMNs cannot have chord-paths with less than 3 edges. Furthermore, we also use a straightforward fact that a SMN must have an even number of vertices of degree 3.

Lemma 2 *Let N be an SMN. If N has two adjacent vertices of degree 3, then N has at least 8 terminals.*

Proof. Assume that N has two adjacent vertices u and v of degree 3. Let a and b denote the other two vertices adjacent to u and let c and d denote the other two vertices adjacent to v. Let C_u denote a cycle through the edges (a,u) and (u,b). Let C_v denote a cycle through the edges (c,v) and (v,d). C_u and C_v must be distinct. If not, (u,v) would be a chord in $C_u = C_v$, contradicting the (length and degree) minimality of N. We claim that C_u and C_v must be disjoint. Assume that this is not the case. Let u' denote the first vertex of C_u encountered when traversing C_v in one direction and let u'' denote the first vertex of C_u when traversing C_u in the other direction. u' and u'' must be distinct; otherwise N would contain a vertex of degree 4, contradicting the minimality of N. The traversed paths from v to respectively u' and u'' together with the portion of C_u between u' and u'' containing u forms a cycle with (u,v) as a chord. This contradicts the minimality of N. From Corollary 1 it follows that N has at least 8 terminals. ■

Lemma 3 *Let N be an SMN. If N has two vertices of degree 3 adjacent to a common terminal of degree 2 then N has at least 9 terminals.*

Proof. The proof is analogous to the proof of Lemma 2. It exploits the fact that N cannot have chord-paths with 2 edges. ■

Lemma 4 *Let N be an SMN. If N has all degree 3 vertices separated by paths with at least 2 terminals of degree 2, then N has at least 6 terminals.*

Proof. Let u and v be two vertices of degree 3 in N. We can always choose u and v such that there is a path P between u and v containing only terminals of degree 2. Let a and b be the vertices adjacent to u not on P. N must contain a cycle C_u through the edges (a,u) and (u,b). C_u avoids any interior terminal of P. By Corollary 1, C_u contains 4 terminals. By assumption, P contains 2 terminals. ■

Lemma 5 *There exist problem instances where SMN has 6 terminals and 2 Steiner vertices.*

Proof. Consider the problem instance shown in Figure 4. It is a complete graph K_8 with 6 terminals (black circles) which have to be spanned and two additional vertices (white circles). All edges shown have unit lengths. All other edges have lengths equal to the lengths of shortest paths through unit edges.

The set of edges shown form a 2-connected subgraph of K_8 spanning all 6 terminals. Hence, it is a feasible 2-connected solution. Its length is 9. We will show that this is the only optimal solution.

Suppose that the solution shown in Figure 4 is not optimal. Assume that an optimal solution contains no vertices of degree 3. Hence it is a traveling salesman tour with 6 edges. It has to contain all three unit length edges (A,D), (B,E) and (C,F) connecting the terminals. Otherwise it would contain 4 or more edges of length at least 2 and would not be optimal. Hence, an optimal solution without degree 3 vertices is forced to contain 3 edges of length 2 connecting pairs of terminals. There are 6 such edges: $(A,B),(A,C),(B,C),(D,E),(D,F),(E,F)$. It can be easily verified by inspection that no three of these edges will together with (A,D), (B,E) and (C,F) form a 2-connected solutions.

The number of vertices of degree 3 in an SMN must be even. If SMN contains one Steiner vertex (and at least one terminal of degree 3), the total number of edges must be at least 8. At least 2 of the edges must have length at least 2. If SMN contains no Steiner vertices (and at least 2 terminals of degree 2), the total number of edges must be at least 7. At least 4 of the edges must have length at least 2. ■

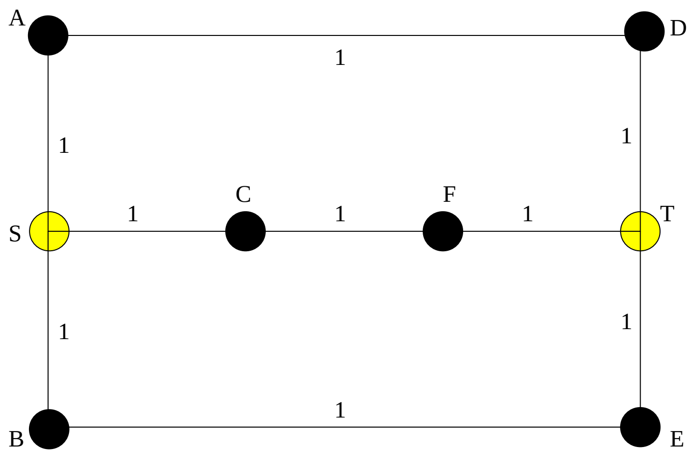


Figure 4: SMN with 2 Steiner vertices and 6 terminals.

We conjecture that in the Euclidean plane with the standard L_2 metric, no problem instance with 7 or less terminals has an SMNP with degree 3 vertices. In particular, such problem instances have no Steiner points and have traveling salesman tours as solutions to 2-SNPP. On the other hand, we have been able to construct a problem instance with 8 terminals which requires two Steiner points. Such an instance is shown in Figure 5.

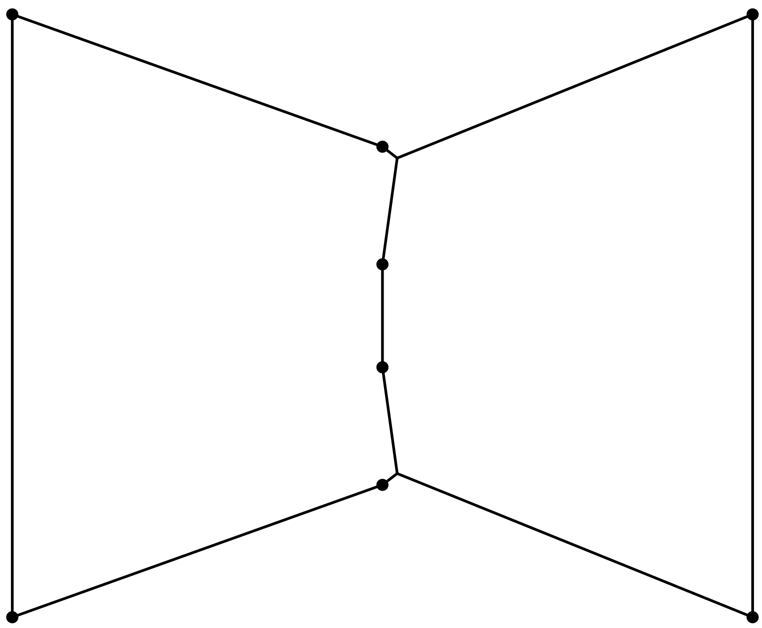


Figure 5: 2-connected SMNP in the Euclidean plane with 8 terminals and 2 Steiner points. This SMNP has two FSTs spanning three terminals and five FSTs spanning two terminals.

6 Conclusion

The structural result presented in this paper have at least two interesting applications. 2-connected SMNs are unions of FSTs. Therefore the general framework for solving the Euclidean, rectilinear [6] and uniformly-oriented [5] Steiner tree problems also applies to 2-connected SMNs. Consequently, the concatenation phase can be made more efficient by adding the constraint that two FSTs can share at most one terminal (which is a direct consequence of Theorem 2).

The other, even more interesting application is when determining good lower bounds for the problem. If we replace every FST in a SMN with a minimum spanning tree for the same set of terminals, this will create a 2-connected network on Z (i.e., without Steiner vertices) in which every edge has multiplicity one. Furthermore, this network fulfills a number of additional properties that can be employed in a new lower bounding scheme and exact algorithm for the problem [7].

References

[1] B. S. Munso C. L. Monma and W. R. Pulleyblank. Minimum-Weight Two-Connected Spanning Networks. *Mathematical Programming*, 46:153–171, 1990.

[2] D. F. Hsu and X.-D. Hu. On Shortest Two-Edge Connected Steiner Networks with Euclidean Distance. *Networks*, 32:133–140, 1998.

[3] E. L. Luebke. *k-Connected Steiner Network Problems*. PhD thesis, University of North Carolina, USA, 2002.

[4] E. L. Luebke and J. S. Provan. On the Structure and Complexity of the 2-Connected Steiner Network Problem in the Plane. *Operations Research Letters*, 26:111–116, 2000.

[5] B. K. Nielsen, P. Winter, and M. Zachariasen. An Exact Algorithm for the Uniformly-Oriented Steiner Tree Problem. In *Proceedings of the 10th European Symposium on Algorithms, Lecture Notes in Computer Science*, volume 2461, pages 760–772. Springer, 2002.

[6] D. M. Warme, P. Winter, and M. Zachariasen. Exact Algorithms for Plane Steiner Tree Problems: A Computational Study. In D.-Z. Du, J. M. Smith, and J. H. Rubinstein, editors, *Advances in Steiner Trees*, pages 81–116. Kluwer Academic Publishers, Boston, 2000.

[7] P. Winter and M. Zachariasen. Two-Connected Steiner Networks: Exact Algorithms. In preparation.